BARKER COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE 1995

MATHEMATICS

AM TUESDAY 22 AUGUST

3/4 UNIT

TIME: 2 HOURS

[Plus 5 minutes' Reading Time

DSK
BJR
JCS
BHC
BTP
JM*

125 copies

DIRECTIONS TO CANDIDATES:

1. **Write your Candidate Number on EVERY page.**

2. **Start each question on a NEW page.**

3. **ALL questions are of equal value.**

4. **Show ALL necessary working. Marks may be deducted for careless or badly arranged work.**

5. **Standard integrals are provided at the end of the paper.**

6. **Board-approved calculators may be used.**

* * * *

QUESTION 1.

(a) Solve for x: $\dfrac{1}{x-1} \leq 3$ [3m]

(b) Find the exact value of $\int_1^{\sqrt{3}} \dfrac{1}{\sqrt{4-x^2}}\,dx$ [2m

(c) Find the derivative of $y = \tan^{-1} 2x$ [2m

(d) Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 2x\,dx$ [3m

(e) (i) Sketch the graph of $y = \sin 2x$ for $0 \leq x \leq 2\pi$ [1m

(ii) Sketch the line $y = \dfrac{1}{2}$. Without solving the equation $\sin 2x = \dfrac{1}{2}$, how many solutions are there for the domain $0 \leq x \leq 2\pi$? [1m

QUESTION 2. [START A NEW PAGE]

(a) $\int_0^1 \frac{x}{1+x}\,dx$ (using the substitution $u = 1 + x$)

(b) Given the function $y = 3\cos^{-1}\left(\frac{x}{2}\right)$:

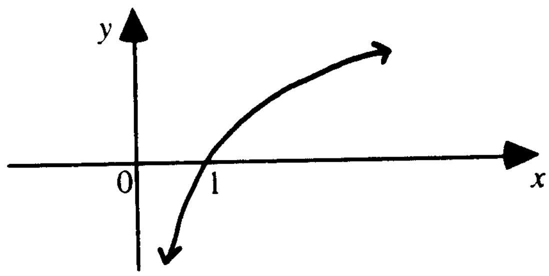
(i) Write down the domain and range.

(ii) Sketch this function.

(c) (i) Express $\sin A$ and $\cos A$ in terms of "t" where $t = \tan\frac{A}{2}$

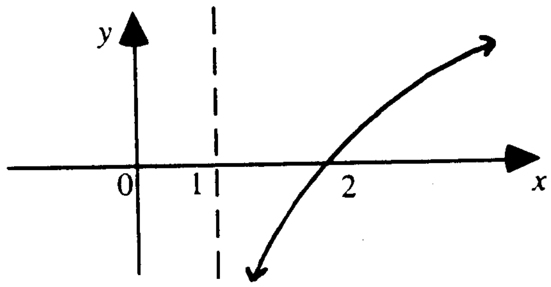
(ii) Hence or otherwise prove that: $\frac{1+\cos 2A}{\sin 2A} = \cot A$

(d) Given that the following is a sketch of $y = \ell n\, x$

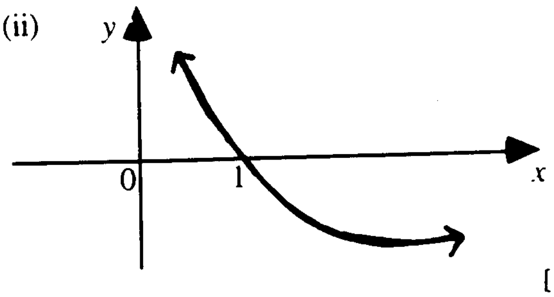


Write down a possible equation for each of the following.

(i)



(ii)



QUESTION 3. [START A NEW PAGE]

(a) Given the function $y = \dfrac{2x+1}{x-1}$

(i) find the domain of this function

(ii) what happens to y when $x \to \infty$

(iii) find any vertical and horizontal asymptotes

(iv) hence sketch (without calculus) a neat graph of the function. [4m]

(b) Find the term independent of x in the expansion $\left(2x^3 - \dfrac{1}{x}\right)^{12}$ [4m]

(c) Solve the trigonometric equation $2\sin^2\theta + \sin^2 2\theta = 2$ for $0 \leq \theta \leq 2\pi$ [4m]

QUESTION 4. [START A NEW PAGE]

(a) Find the acute angle between $2x - y + 5 = 0$ and $y = -3x + 7$ [3m]

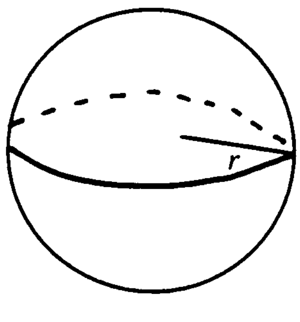
(b) The velocity $v\ m/s$ of a point moving along the x-axis is given by $v^2 = 16x - 4x^2 + 20$

(i) Prove that the motion is simple harmonic

(ii) Find the centre of motion

(iii) Find the length of the path. [4m]

(c)



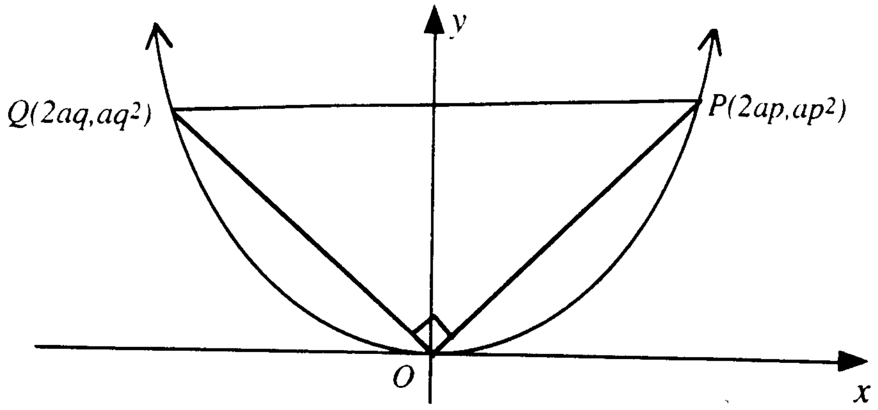
$$V = \frac{4}{3}\pi r^3$$
$$A = 4\pi r^2$$

A spherical balloon is being inflated and its volume increases at a constant rate of 50 mm^3 per second.

At what rate is its surface area increasing when the radius is 20 mm? [5m

QUESTION 5. [START A NEW PAGE]

(a)



PQ is a variable chord of the parabola $x^2 = 4ay$.

It subtends a right angle at the vertex O

If p and q are the parameters corresponding to the points P, Q respectively :

(i) Show that the equation of the tangent to $x^2 = 4ay$ at P is $y - px + ap^2 = 0$

(ii) Hence write down the equation of the tangent at Q, and then find R, the point of intersection of the two tangents drawn from P and Q.

(iii) Find the gradients of PO and QO and hence prove $pq = -4$

(iv) Show that the locus of this point of intersection is $y = -4a$ [7m

(b) Use mathematical induction to prove that for all positive integers:

$$1^2 + 2^2 + 3^2 + \dots\dots\dots\dots\dots\dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$ [5

QUESTION 6. [START A NEW PAGE]

(a) A touring cricket side of 15 players contains 5 regular bowlers.

(i) How many different elevens can be picked which contain exactly 3 of the 5 regular bowlers?

(ii) What is the probability that if an eleven is picked at random it will only contain 1 regular bowler?

(iii) What is the probability that if an eleven is picked at random it will contain at least 3 of the regular bowlers? [5m

(b) (i) Write down the expansion for $(1+x)^n$

(ii) Using this expansion, show that:

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots\dots\dots\dots = 2^{n-1}$$

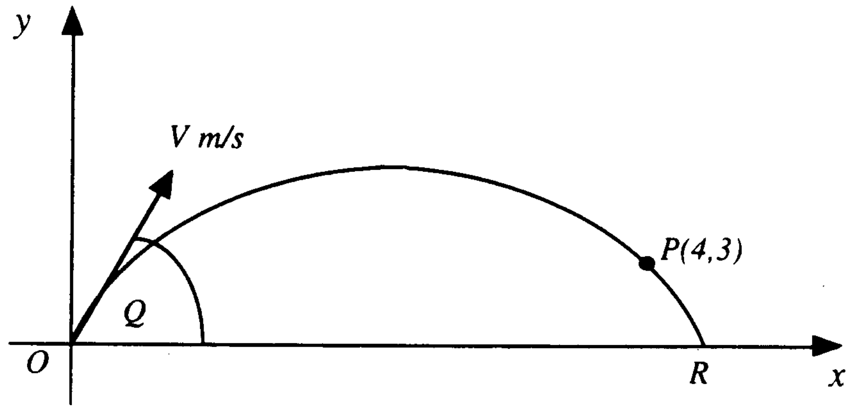
(iii) From the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$ compare the coefficient of x^{n+1} on both sides and hence prove that:

$$\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + \dots\dots\dots\dots + \binom{n}{n-1}\binom{n}{n} = \frac{(2n)!}{(n-1)!\,(n+1)!}$$ [7

QUESTION 7. [START A NEW PAGE]

(a) How many *"words"* can be formed from AUSTRALIA? (taken all at a time) [2

(b) A particle is projected from a point O with a speed of $V\,m/s$ at an angle of θ to the horizontal. Air resistance is to be neglected and $g\,m/s^2$ is the acceleration due to gravity.



(i) Starting from $\ddot{x} = 0$ show that $x = Vt\cos\theta$ [2

(ii) Starting from $\ddot{y} = -g$ show that $y = \frac{-1}{2}gt^2 + Vt\sin\theta$ [2

(iii) Prove that the Cartesian equation of path of projectile is given by:

$$y = \frac{-gx^2}{2V^2}\left(1+\tan^2\theta\right) + x\tan\theta$$ [

(iv) You are given that $V^2 = 8g$ and that the particle passes through a point $P(4,\ 3)$. Hence by using the equation in (iii) find θ, the initial angle of projection and R, the range of the projectile. [

E N D O F P A P E R

Year 12 3 unit Trial HSC 1995 DSK

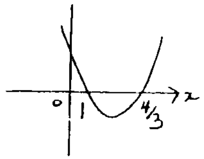
Q1. a) $\frac{1}{x-1} \le 3$

$x-1 \le 3(x-1)^2$ (1)

$0 \le 3(x-1)^2 - (x-1)$

$0 \le (x-1)[3(x-1)-1]$

$0 \le (x-1)(3x-4)$ (1)



$\therefore x \le 1,$

$x \ge \frac{4}{3}.$ (1)

b) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$ (1)

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$ (1)

c) $y = \tan^{-1} 2x$

$y' = \frac{1}{1+4x^2} \cdot 2 = \frac{2}{1+4x^2}$

↑ (1) ↑ (1)

d) $\int_0^{\frac{\pi}{6}} \sin^2 2x \, dx$

$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 4x)\,dx$ (1)

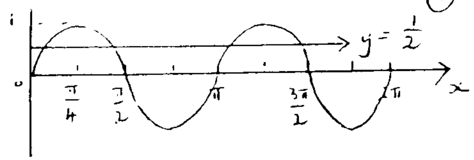
$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/6}$

$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right]$ (1)

$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right]$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{16} = \frac{4\pi - 3\sqrt{3}}{48}.$ (1)

e) i) $y = \sin 2x$ (1)



ii) 4 solutions. (1)

Q2. a) $\int_0^1 \frac{x}{1+x}\,dx$

Let $u = 1+x$

$\frac{du}{dx} = 1$

$du = dx$

when $x = 1,\ u = 2$

when $x = 0,\ u = 1$

$\int_0^1 \frac{x}{1+x}\,dx$

$= \int_1^2 \frac{u-1}{u}\,du$ (1)

$= \int_1^2 \left(1 - \frac{1}{u}\right)du$

$= \left[u - \log u\right]_1^2$ (1)

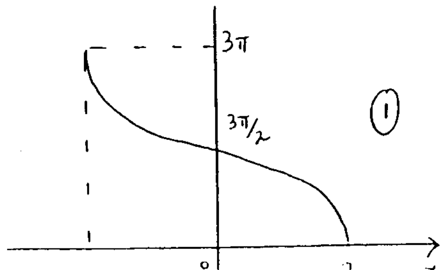
$= 2 - \log 2 - 1 + \log 1$

$= 1 - \log 2.$ (1)

b) i) Domain is $-2 \le x \le 2$ (1)

Range is $0 \le y \le 3\pi.$ (1)

ii) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$



(1)

c) i) If $t = \tan\frac{A}{2}$ then

$\sin A = \frac{2t}{1+t^2}$ (1)

$\cos A = \frac{1-t^2}{1+t^2}$ (1)

ii) Prove $\frac{1+\cos 2A}{\sin 2A} = \cot A.$

$LHS = \frac{1+(2\cos^2 A - 1)}{2\sin A\cos A}$

$= \frac{2\cos^2 A}{2\sin A\cos A}$

$= \frac{\cos A}{\sin A} = \cot A = RHS.$

OR :

$LHS = \frac{1 + \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$

$= \frac{(1+t^2)+(1-t^2)}{2t}$ (1)

$= \frac{1}{t} = \frac{1}{\tan A} = \cot A = RHS.$ (1)

d) i) $y = \log_e(x-1)$ (1)

ii) $y = -\log_e x$ (1)

$x - 1$

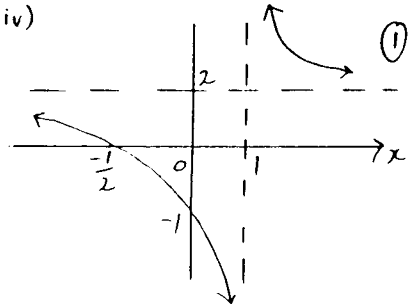
i) Domain is all x, $x \neq 1$. (1)

ii) As $x \to \infty$, $y \to 2$. (1)

iii) Vertical asy. at $x = 1$. (1)

Horiz. asy. at $y = 2$.

iv) (1)



b) The rth term in the expansion of $(2x^3 - \frac{1}{x})^{12}$ is

$${}^{12}C_{r-1}\,(2x^3)^{13-r}\left(\frac{-1}{x}\right)^{r-1} \quad (1)$$

$$= {}^{12}C_{r-1}\cdot(-1)^{r-1}\cdot(2)^{13-r}\cdot x^{39-3r+1-r} \quad (1)$$

$$= {}^{12}C_{r-1}\,(-1)^{r-1}\cdot(2)^{13-r}\cdot x^{40-4r}$$

(1)

This is independent of x if $r = 10$

ie, u_{10} is indep. of x. It is

$${}^{12}C_9\cdot(-1)^9\cdot 2^3$$

$$= -1760 \quad (1)$$

$$2(1-\cos^2\theta) + \sin^2 2\theta = 2 \quad (1)$$

$$2 - 2\cos^2\theta + \sin^2 2\theta = 2$$

$$\sin^2 2\theta - 2\cos^2\theta = 0 \quad (1)$$

$$(\sin 2\theta - \sqrt{2}\cos\theta)(\sin 2\theta + \sqrt{2}\cos\theta)$$

$$\cos^2\theta(2\sin\theta - \sqrt{2})(2\sin\theta + \sqrt{2})$$

$$\cos\theta = 0 \quad \text{OR} \quad \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$$

(1)

has $m_1 = 2$. (1)

$y = -3x + 7$ has $m_2 = -3$.

Let α be angle between these lines, then

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{2 - -3}{1 + 2(-3)}\right| = 1 \quad (1)$$

$\therefore \alpha = 45°$. (1)

b) $v^2 = 16x - 4x^2 + 20$

i) $\frac{1}{2}v^2 = 8x - 2x^2 + 10$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 8 - 4x \quad (1)$$

$$= -4(x-2)$$

$$= -2^2(x-2). \quad (1)$$

Since $\ddot{x}$ is of the form $\ddot{x} = -n^2 X$, $X = x - 2$, then motion is SHM.

ii) Centre is $x = 2$. (1)

iii) Length of path is 6 units. (1)

$$A = 4\pi R^2$$

$$\frac{dA}{dR} = 8\pi R$$

$$\frac{dA}{dt} = \frac{dA}{dR}\cdot\frac{dR}{dt}$$

$$= 8\pi R\cdot\frac{dR}{dt} \quad (1)$$

To find $\frac{dR}{dt}$:

$$V = \frac{4}{3}\pi R^3$$

$$\frac{dV}{dR} = 4\pi R^2$$

$$\frac{dV}{dt} = \frac{dV}{dR}\cdot\frac{dR}{dt} \quad (1)$$

$$50 = 4\pi R^2\cdot\frac{dR}{dt}$$

$$\frac{dR}{dt} = \frac{50}{4\pi R^2} = \frac{25}{2\pi R^2} \quad (1)$$

Hence

$$\frac{dA}{dt} = 8\pi R\cdot\frac{25}{2\pi R^2} \quad (1)$$

$$= \frac{100}{R}$$

When $R = 20$,

$$\frac{dA}{dt} = 5\ \text{mm}^2/\text{sec}. \quad (1)$$

Q5. a)

→ iii) m of $OQ = \dfrac{aq^2 - 0}{2aq - 0}$

$= \dfrac{q}{2}$

m of $OP = \dfrac{ap^2 - 0}{2ap - 0}$ (1)

$= \dfrac{p}{2}$

Since $Q\hat{O}P = 90^\circ$ then

$\dfrac{q}{2} \times \dfrac{p}{2} = -1$

ie, $pq = -4$ (1)

→ i) $y = \dfrac{1}{4a} x^2$

$f'(x) = \dfrac{x}{2a}$

$f'(2ap) = \dfrac{2ap}{2a} = p.$

Hence eq'n of tangent at P is (1)

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

ie, $y - px + ap^2 = 0$ (1)

→ ii) Solve $\begin{cases} y - px + ap^2 = 0 \\ y - qx + aq^2 = 0. \end{cases}$

(1) − (2): $-px + qx + ap^2 - aq^2 = 0$ (1)

$x(q - p) = a(q^2 - p^2)$ (1)

ie, $x = a(q + p)$

$\therefore \quad y = apq$ (1)

R is $\{a(p+q),\ apq\}$

iv) $x = a(p+q)$

$y = apq$

But $pq = -4$, so

$y = -4a$

is locus of R. (1)

b) STEP 1 When $n = 1$,

$LHS = 1^2 = 1$

$RHS = \frac{1}{6} \cdot 1(2)(3) = 1.$ (1)

Hence the proposition is true when $n = 1$.

STEP 2 We assume that

$1^2 + 2^2 + \dots + k^2 = \frac{1}{6} k(k+1)(2k+$ …

is true, and now show that

$1^2 + 2^2 + \dots + k^2 + (k+1)^2$ (1)

$= \frac{1}{6}(k+1)(k+2)(2k+3).$

Now $LHS = \frac{1}{6} k(k+1)(2k+1) + (k+$ … (1)

$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+\right.$ …

$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$ (1)

$= RHS.$

STEP 3. We have shown that the prop. is true when $n=1$. Assuming that the prop true when $n = k$, we have shown that prop. is true for $n = k+1$ also. Hence the prop. is true when $n = 2$, and so true for $n = 3$ and so on. Hence it's true for all natural numbers (1)

Q6. a)

i) ${}^{5}C_3 \times {}^{10}C_8 = 450$ (1)

ii) With no restrictions, there are

${}^{15}C_{11} = 1365$ elevens. (1)

The number with only one regular bowler is

${}^{5}C_1 \times {}^{10}C_{10} = 5.$

Hence required probability is

$\dfrac{5}{1365} = \dfrac{1}{273}.$ (1)

iii) Probability (1)

$= \dfrac{{}^{5}C_3 \times {}^{10}C_8 + {}^{5}C_4 \times {}^{10}C_7 + {}^{5}C_5 \times {}^{10}C_6}{1365}$

$= \dfrac{1260}{1365} = \dfrac{12}{13}$ (1)

b) i) $(1 + x)^n$ (1)

$= {}^{n}C_0 x^0 + {}^{n}C_1 x^1 + \dots + {}^{n}C_n x^n.$

ii) Let $x = 1$, then

$LHS = 2^n$

$RHS = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

(1)

Next, let $x = -1$, then

$LHS = 0$ (1)

$RHS = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \dots$

ie, $\binom{n}{1} + \binom{n}{3} + \dots = \binom{n}{0} + \binom{n}{2} + \dots$

ie, $\binom{n}{1} + \binom{n}{3} + \dots = \frac{1}{2} \times 2^n$

$= 2^{-1} \times 2^n$ (1)

$= 2^{n-1}$ as required

(iii) Consider $(1+x)^n (1+x)^n = (1+x)$ …

Now the coeff. of x^{n+1} is

$\binom{n}{1}\binom{n}{n} + \binom{n}{2}\binom{n}{n-1} + \binom{n}{3}\binom{n}{n-2} + \dots$

$+ \binom{n}{n}\binom{n}{1}$ (1)

$= \binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \dots + \binom{n}{n-1}($ …

Considering the coeff of x^{n+1} from $(1+x)^{2n}$ we get

${}^{2n}C_{n+1}\, x^{n+1} = \dfrac{(2n)!}{(n+1)!(2n-(n+1))!} x^{n+1}$ (1)

Hence

$\binom{n}{0}\binom{n}{1} + \dots + \binom{n}{n-1}\binom{n}{n} = \dfrac{(2n)!}{(n+1)!(n-1)!}$

as required. (1)

Q. a) $\frac{\quad}{3!} = 60480$ ← (1)

b) i) $\ddot{x} = 0$

$\dot{x} = c$

But when $t = 0$, $\dot{x} = V\cos\theta$

so $c = V\cos\theta$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + c_2$

But $x = 0$ when $t = 0$ so $c_2 = 0$

ie, $x = Vt\cos\theta$ (2)

ii) $\ddot{y} = -g$

$\dot{y} = -gt + c_3$

But when $t = 0$, $\dot{y} = V\sin\theta$

so $c_3 = V\sin\theta$ so

$\dot{y} = -gt + V\sin\theta$

$y = -\frac{gt^2}{2} + Vt\sin\theta + c_4$

But $y = 0$ when $t = 0$ so $c_4 = 0$

ie, $y = -\frac{gt^2}{2} + Vt\sin\theta$ (2)

iii) From $x = \ldots$

$t = \frac{x}{V\cos\theta}$ so,

by substitution,

$y = -\frac{g}{2}\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$

$= \frac{-gx^2}{2V^2} \cdot \frac{1}{\cos^2\theta} + \frac{x\sin\theta}{\cos\theta}$

$= -\frac{gx^2}{2V^2} \cdot \sec^2\theta + x\tan\theta$

$= -\frac{gx^2}{2V^2}(1 + \tan^2\theta) + x\tan\theta$ (2)

as required.

iv) From (iii)

$3 = \frac{-g \cdot 4^2}{2 \cdot 8g}(1 + \tan^2\theta) + 4\tan\theta$

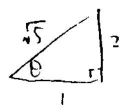
$3 = -1(1 + \tan^2\theta) + 4\tan\theta$ (1)

$\tan^2\theta - 4\tan\theta + 4 = 0$

$(\tan\theta - 2)(\tan\theta - 2) = 0$

$\tan\theta = 2$

$\theta = 63°26'$ (1)



To find Range, solve $y = 0$ firstly.

$0 = \frac{-gt^2}{2} + Vt\sin\theta$

$\frac{gt^2}{2} = Vt\sin\theta$

$t = \frac{2V\sin\theta}{g}$ (1)

Sub this into $x = Vt\cos\theta$ and we get

$R = x = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$

$= \frac{V^2 2\sin\theta\cos\theta}{g}$

$= \frac{V^2\sin 2\theta}{g}$ (1)

$= \frac{8g\sin 2\theta}{g}$

$= 8 \cdot 2\sin\theta\cos\theta$

$= 16 \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{32}{5}$ or 6.4 m